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The transient retardation of a rectilinear viscoplastic flow when the loading stresses are abruptly removed☆

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ABSTRACT

The development of a viscoplastic flow in a solid layer of an elastoviscoplastic material on an inclined plane is considered when loading stresses act on its free surface. It is shown that the elastoplastic boundary starts its motion from the rigid inclined plane and, propagating through the elastic core, it can reach the free surface of the layer. An exact solution is obtained for the dynamic problem of the retardation of developed viscoplastic flow after the loading stresses are abruptly removed. The possibility of writing the equation of motion for the unloading wave in terms of the displacements is pointed out. It reduces to an inhomogeneous wave equation where the velocity of the unloading wave is found to be equal to the velocity of the equivoluminal elastic wave. Reflection of the unloading wave from a rigid boundary in the form of an inclined plane is also considered.

The surfaces of strain discontinuities in elastoplastic media separate into loading and unloading waves. A review of papers, dealing with the distinctive features of the propagation of such discontinuities can be found in Ref. 1. The conditions for the existence of surfaces of strain discontinuities in media with elastic and plastic properties and the regularities in their propagation were studied later.[2–6] When the viscous properties of the medium are taken into account in the plastic flow process, no other discontinuity surfaces, apart from those which propagate with the known velocities of elastic waves, develop;[7,8] this is also recorded in experiments.[2] In the case of the propagation of unloading waves, the situation is further complicated by the fact that the discontinuity surface propagates through a viscoplastic flow domain where the irreversible deformations cannot be considered to be small.

Here, we consider the simplest boundary value problem in the theory of large elastoviscoplastic deformations with an unloading wave. Such a plane of stress discontinuity arises in a solid layer of an incompressible elastoviscoplastic material on an inclined plane, and the viscoplastic flow developed in this layer is caused by the action of the loading stresses on the free surface of the layer after they are abruptly removed. The principal difficulty in formulatiing of the corresponding boundary-value problem is associated with the fact that the plane of stress discontinuity moves through an intensively and irreversibly deforming medium while the stresses are determined by the level and distribution of the reversible strains. Hence, in order to determine the dependence of the stresses on the displacements and to write down the equation of motion of the medium, it is necessary to specify the distributions of the plastic and elastic strains beyond the unloading wave. Moreover, when writing the equation of motion in the displacements, it is also necessary to find the displacements in accordance with the definition of reversible and irreversible strains which, as is well-known,[9] is not a simple problem even in the case of small strains. It is obvious that assuming that the material is incompressible both during the irreversible strain stage as well as the reversible strain stage assists in overcoming the mathematical difficulties. Taking account of bulk gravitational forces is of fundamental importance for the correct formulation of the problem.[10]

1. Initial relations for the strain model used

In constructing a flow theory taking account of large elastoplastic strains, it is necessary to determine the reversible and irreversible strains, which are not measurable experimentally, as components of the overall strains following the additional assumptions in Refs 11–13. In non-equilibrium thermodynamics, reversible and irreversible strains relate to thermodynamic parameters and their determination is

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therefore associated with the postulation of variation (transport) equations for them.[14–16] One of the basic assumptions of the model constructed[15–17] is the requirement that irreversible strains in unloading processes are invariant and, also that of the thermodynamic potential (the internal energy and free energy) is independent of the irreversible strains which enables not only the basic relations of the model to be written in a simple form but also enables, within its limits, a number of boundary value problems concerning quasistatic irreversible deformation with subsequent unloading to the solved.[10,18,19]

Here, we consider the simplest problem in the strain dynamics of an elastoviscoplastic medium with initial conditions which are attained due to quasistatic strain. Using the approach described earlier,[15,17] we specify the strain kinematics in a rectangular system of Euler coordinates x_i using the relations

$$d_{ij} = e_{ij} + p_{ij} - \frac{1}{2}e_{ik}e_{kj} - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{km}e_{mj}$$

$$\frac{De_{ij}}{Dt} = \varepsilon_{ij} - \varepsilon_{ij}^p - \frac{1}{2}((\varepsilon_{ik} - \varepsilon_{ik}^p + z_{ik})e_{kj} + e_{ik}(\varepsilon_{kj} - \varepsilon_{kj}^p - z_{kj}))$$

$$\frac{Dp_{ij}}{Dt} = \varepsilon_{ij}^p - p_{ik}\varepsilon_{kj}^p - \varepsilon_{ik}^p p_{kj}, \quad \frac{Dn_{ij}}{Dt} = \frac{dn_{ij}}{dt} - r_{ik}n_{kj} + n_{ik}r_{kj}$$

$$\varepsilon_{ij} = \frac{1}{2}(\upsilon_{i,j} + \upsilon_{j,i}), \ \upsilon_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_{i,j}\upsilon_j, \ u_{i,j} = \frac{\partial u_i}{\partial x_j}, \ r_{ij} = \frac{1}{2}(\upsilon_{i,j} - \upsilon_{j,i}) + z_{ij}(\varepsilon_{sk}, e_{sk}) \tag{1.1}$$

In relations (1.1), d_{ij} are the components of the Almans Almansi strains, e_{ij} and p_{ij} are their reversible and irreversible components, D/Dt is an objective derivative of the tensors with respect to time, written for an arbitrary tensor with the components n_{ij}, u_i and υ_i, that is, the components of the displacement vector and the velocity vector of the points of the medium, ε_{ij}^p are the components of the plastic strain rate tensor, and the non-linear component z_{ij} of the rotation tensor r_{ij} has been written out fully in Ref. 15.

As consequence of the law of conservation of energy, assuming that the thermodynamic potentials are independent of the irreversible strain the Murnaghan formulae are

$$\sigma_{ij} = \begin{cases} -p\delta_{ij} + \dfrac{\partial W}{\partial d_{ik}}(\delta_{ik} - 2d_{kj}) \text{ when } p_{ij} \equiv 0 \\ -p_1\delta_{ij} + \dfrac{\partial W}{\partial e_{ik}}(\delta_{kj} - e_{kj}) \text{ when } p_{ij} \neq 0 \end{cases}$$

$$W = -2\mu J_1 - \mu J_2 + bJ_1^2 + (b-\mu)J_1J_2 - \chi J_1^3 + \dots, \quad J_k = \begin{cases} L_k \text{ when } p_{ij} = 0 \\ I_k \text{ when } p_{ij} \neq 0 \end{cases}$$

$$L_1 = d_{kk}, \quad L_2 = d_{ik}d_{ki}, \quad I_1 = e_{kk} - \frac{1}{2}e_{sk}e_{ks}, \quad I_2 = e_{st}e_{ts} - e_{sk}e_{kr}e_{ts} + \frac{1}{4}e_{sk}e_{kr}e_{tn}e_{ns} \tag{1.2}$$

In relations (1.2), σ_{ij} are the components of the stress tensor, p and p_1 are the additional hydrostatic pressures, W is the elastic potential and μ, b and χ are constants of the material.

We will assume that the irreversible strains in the material accumulate when the stress state is attaining the loading surface which, according to the accepted Mises maximum principle, is the plastic potential. We shall specify this surface by the Tresca yield criterion extended to the case of viscoplastic flow:

$$\max|\sigma_i - \sigma_j| = 2k + 2\eta\max|\varepsilon_k^p| \tag{1.3}$$

In relation (1.3), σ_{ij} and ε_k^p are the principal values of the stress and plastic strain rate tensors, k is the yield point and η is the coefficient of viscosity.

The relation between the irreversible strain rates and the stresses is established by the associated plastic flow law

$$\varepsilon_{ij}^p = \lambda\frac{\partial f}{\partial \sigma_{ij}}, \quad f(\sigma_{ij}, \varepsilon_{ij}^p) = k, \quad \lambda > 0 \tag{1.4}$$

2. Quasistatic viscoplastic flow

Consider the linear motion of an elastoviscoplastic medium constituting a layer of solid material on an inclined plane and loaded on its free surface. The coordinate axes are chosen such that the x_2 axis is directed downwards along the loaded surface of the layer and the x_1 axis is directed into the layer. We initially assume that the material is in equilibrium, with the boundary conditions

$$u|_{x_1 = h} = 0, \quad \sigma_{11}|_{x_1 = 0} = -\sigma, \quad \sigma_{12}|_{x_1 = 0} = \xi \tag{2.1}$$

Here, $u = u_2(x_1)$ is the only non-zero component of the displacement vector, h is the layer thickness, and σ and ξ are given constants. The constant σ can be arbitrary (in particular, it can be equated to zero). The values of the constant ξ are limited by the conditions for a stress state to occur on the loading surface (1.3). If $\xi_* \le \xi < 0$ (a shift down along the inclined plane), the plasticity condition is satisfied in the plane $x_1 = h$ in the form

$$\sigma_{12}|_{x_1 = h} = -k \tag{2.2}$$

In the case when $0 < \xi \le \xi^*$ (a shift upwards along the inclined plane) the plasticity condition $\sigma_{12}|_{x_1=0} = k$ is satisfied in the plane $x_1 = 0$. The problem is solved for the first case.

The stress-strain state parameters at the instant when plasticity condition (2.2) is satisfied are calculated by integrating the equilibrium equations using conditions (2.1) and (2.2), and, at the same time, account is taken of the fact that the stress components are related to the overall strain tensor components by the first formula of (1.2). We have

$$\sigma_{11} = \sigma_{33} = -\rho g_1 x_1 - \sigma, \quad \sigma_{12} = \rho g_2(h - x_1) - k$$

$$\sigma_{22} = -\rho g_1 x_1 - \sigma + \frac{1}{\mu}(\rho g_2(h - x_1) - k)^2, \quad u = \frac{1}{2\mu}(-\varepsilon g_2(x_1 - h)^2 + 2k(h - x_1))$$

$$\xi_* = \rho g_2 h - k, \quad g_1 = g\cos\phi, \quad g_2 = g\sin\phi \tag{2.3}$$

Here, ρ is the density of the material, g is acceleration due to gravity and φ is the angle of inclination of the plane.

Henceforth, only the leading non-linear terms are left in relations (2.3). These relations serve as initial conditions for the subsequent plastic flow as the loading forces increase with time

$$\sigma_{11}|_{x_1 = 0} = -c_2(t), \quad \sigma_{12}|_{x_1 = 0} = c_1(t) \tag{2.4}$$

The function $c_2(t)$, as well as the constant σ, can be arbitrary since σ_{11} has no effect on the plastic flow. At the same time, $c_1(0) = \xi_*$.

From the instant $t = 0$, the viscoplastic flow domain is bounded by the planes $m(t) \le x_1 \le h$ and the elastic core occupies the domain $0 \le x_1 \le m(t)$. The surface $m(t)$ is the moving boundary of the viscoplastic flow domain. The stress-strain state parameters at any instant $t = t_1 > 0$ are found by integrating the equilibrium equations using boundary conditions (2.1) and (2.4) and the plasticity condition in the form

$$\sigma_{12} = -k + \eta\varepsilon_{12}^p \tag{2.5}$$

The conditions for the stress components, the strain components and the derivatives $\partial u/\partial x_1$ and $\partial u/\partial t = \upsilon$ to be equal are satisfied in the elastoplastic boundary $x_1 = m(t)$ which enable as to find the integration functions. Hence, the following relations are the solution of the problem:

in the domain of reversible strain $0 \le x_1 \le m(t)$

$$u = \rho g_2 t\eta^{-1}(m - h)^2 + F, \quad F = \frac{1}{2\mu}(\rho g_2(h^2 - x_1^2) - 2c_1(t_1)(h - x_1))$$

and in the domain of viscoplastic flow $m(t) \le x_1 \le h$

$$u = \rho g_2 t\eta^{-1}(h^2 - x_1^2 + 2m(x_1 - h)) + F$$

$$p_{11} = -2p_{12}(e_{12} - p_{12}), \quad p_{12} = \rho g_2 t\eta^{-1}(m - x_1), \quad p_{22} = 2e_{12}p_{12} \tag{2.6}$$

The value of $m_1 = m(t_1)$, which determines the elastoplastic boundary and corresponds to the applied load $c_1(t_1)$, is determined from the condition for the plastic strain rates ε_{12}^p (2.5) to be equal to zero when $x_1 = m_1$, where

$$m_1 = (c_1(t_1) + k)/(\rho g_2) \tag{2.7}$$

The stresses in the two domains are determined by the relations

$$\sigma_{11} = \sigma_{33} = -\rho g_1 x_1 + c_2(t_1), \quad \sigma_{12} = -\rho g_2 x_1 + c_1(t_1)$$

$$\sigma_{22} = -\rho g_1 x_1 + c_2(t_1) + (c_1(t_1) - \rho g_2 x_1)^2\mu^{-1} \tag{2.8}$$

3. Unloading dynamics

Starting from a certain instant $t \ge t_2$ in the case of a loading $c_1(t) \le -k$, viscoplastic flow of the whole of the layer of the material occurs. We will assume that, at some subsequent instant $t = t^*$, the load $c_1(t^*) = -k_0$ is abruptly removed:

$$\sigma_{12}(x_1 = 0, t = t^*) = 0 \tag{3.1}$$

Hence, starting from the instant $t = t^*$, the surface $x_1 = G(t - t^*)$ moves from the plane $x_1 = 0$ to the plane $x_1 = h$ (it is shown below that $G = \sqrt{\mu/\rho}$) which singles out the unloading zone

$$0 \le x_1 \le G(t - t^*) \tag{3.2}$$

from the domain of continuing plastic flow

$$G(t - t^*) \le x_1 \le h \tag{3.3}$$

According to the transport equation for the irreversible strain tensor (1.1), under unloading $\varepsilon_{ij}^p = 0$ the components of the irreversible strain tensor change as in the case of the rigid displacement of a body. It follows from the kinematic relations

$$\varepsilon_{11}^p = \frac{dp_{11}}{dt} + 2p_{12}(r_{21} + \varepsilon_{12}^p), \quad \varepsilon_{12}^p = \frac{dp_{12}}{dt} = \frac{\partial p_{12}}{\partial t}, \quad \varepsilon_{22}^p = \frac{dp_{22}}{dt} + 2p_{12}(r_{12} + \varepsilon_{12}^p)$$

which hold in the case being considered, that the irreversible components straing p_{12} do not change in the unloading zone. According to the penultimate relation of (2.6), the distribution of p_{12} through the layer in the plastic flow domain at an instant $t \geq t^*$ is given by the relation

$$p_{12} = -\rho g_2 t \eta^{-1} x_1 \tag{3.4}$$

Subsequently, p_{12} changes at each point up to the instant when the surface $x_1 = G(t-t^*)$ reaches it, and, from this instant, it does not vary in the domain (3.2). Hence, at any instant in the unloading zone (3.2), p_{12} is a function of only the coordinate x_1 and it is time-independent.

It follows from formula (3.4) that, in the unloading zone,

$$p_{12} = -\rho g_2 \eta^{-1}(x_1^2 G^{-1} + t^* x_1) \tag{3.5}$$

When account is taken of the relation $\sigma_{12} = \mu(\partial u/\partial x_1 - 2p_{12})$, from the equation of motion of the medium we derive the equation for finding the displacement components

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{G^2}\frac{\partial^2 u}{\partial t^2} = -\rho g_2\left(\frac{2t^*}{\eta} + \frac{1}{\mu} + \frac{4x_1}{\eta G}\right) \tag{3.6}$$

According to equality (3.1) and the requirement that the displacement components should be continuous on the boundary $x_1 = G(t-t^*)$, the boundary conditions for equation (3.6) are

$$\left.\frac{\partial u}{\partial x_1}\right|_{x_1 = 0} = 0, \quad u|_{x_1 = G(t-t^*)} = \rho g_2 t \eta^{-1}(h^2 - x_1^2) + F \tag{3.7}$$

The solution of Eq. (3.6) has the form

$$u = \frac{\rho g_2 G^2}{4}\alpha\beta\left(\frac{2t^*}{\eta} + \frac{1}{\mu} + \frac{\beta - \alpha}{\eta}\right) + f(\alpha) + g(\beta); \quad \alpha = t - \frac{x_1}{G}, \quad \beta = t + \frac{x_1}{G} \tag{3.8}$$

Using boundary conditions (3.7) for finding the functions $f(\alpha)$ and $g(\beta)$ and returning to the initial variables, we finally obtain

$$u = \frac{k_0}{\mu}(h - G(t^* - t)) + \frac{\rho g_2}{2\mu}(h^2 - x_1^2) + \frac{\rho g_2}{2\eta}\left(2h^2 t - \frac{4}{3G}x_1^3 + \frac{G^2}{6}(t - t^*)^3 - \frac{x_1^2}{2}(3t^* + t)\right)$$
$$\sigma_{12} = -\rho g_2 x_1\left(1 + \frac{\mu}{2\eta}(t - t^*)\right) \tag{3.9}$$

According to (2.6) and (2.7), the relations

$$u = \rho g_2(h^2 - x_1^2)\left(\frac{t}{\eta} + \frac{1}{2\mu}\right) - \frac{k_0}{\mu}(x_1 - h), \quad \sigma_{12} = -\rho g_2 x_1 - k_0$$

hold in the domain of continuing plastic flow (3.3).

4. Reflection of on unloading wave from a rigid wall

At the instant $t = t_3$, the surface $x_1 = G(t-t^*)$ reaches the fixed plane $x_1 = h$. Consequently, starting from the instant $t = t_3$, the surface $x_1 = h - G(t-t_3)$ moves from the plane $x_1 = h$ to the plane $x_1 = 0$. In order to find the components of the displacement u and the stress σ_{12}, we use solution (3.8) of Eq. (3.6), obtained earlier. The first condition of (2.1) and the condition

$$u|_{x_1 = h - G(t-t_3)} = \frac{k_0}{\mu}G(t_3 - t) + \frac{\rho g_2}{2\mu}G(t_3 - t)(2h - G(t_3 - t)) +$$
$$+ \frac{\rho g_2}{\eta}\left(G^2\left(\frac{1}{3}(t^3 - 4t_3^3) + t_3 t(3t_3 - 2t)\right) + h^2(t - t_3) + Gh(4t_3 t - 3t_3^2 - t^2)\right) \tag{4.1}$$

will now be the boundary conditions for this equation.

Relation (4.1) follows from formula (3.9), and it is a consequence of the continuity of the displacement components in the surface $x_1 = h - G(t-t_3)$.

As was done earlier, using the boundary conditions to find the functions $f(\alpha)$ and $g(\beta)$, we obtain the solution of this boundary-value problem in the domain $h-G(t-t_3)\le x_1\le h$ in the form

$$u = \frac{k_0}{\mu}(x_1-h)+\frac{\rho g_2}{2\mu}(h^2-x_1^2)+$$

$$+\frac{\rho g_2}{4\eta}\left[G(x_1-h)(t-t_3)^2+\frac{3x_1h}{G}(x_1-h)+4t_3(h^2-x_1^2)+\frac{7}{3G}(h^3-x_1^3)\right]$$

$$\sigma_{12} = k_0-\rho g_2x_1+\frac{\rho g_2\mu}{4\eta}\left[G(t-t_3)^2+\frac{1}{G}(x_1^2-3h^2-2hx_1)\right]$$

Relations (3.9) are the solution in the domain $0\le x_1\le h-G(t-t_3)$.

The assumption that the irreversible strain tensor is invariant when the load is removed is therefore the fundamental requirement of the model, which enables as to construct a closed solution. It is precisely this that enabled us to write down the inhomogeneous wave equation (3.6) as a consequence of the equation of motion. In the two solutions obtained, the surfaces $x_1=G(t-t^*)$ and $x_1=h-G(t-t_3)$ are surfaces of discontinuity in the stresses (elastic strains) and the overall deformations.

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